Is hyper-extensionality preservable under deletions of graph elements?*

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Abstract. Any hereditarily finite set S can be represented as a finite pointed graph –dubbed membership graph– whose nodes denote elements of the transitive closure of $\{S\}$ and whose edges model the membership relation. Membership graphs must be hyper-extensional –nodes are pairwise not bisimilar– and bisimilar nodes represent the same hereditarily finite set. It is worth to notice that the removal of even a single node or edge from a membership graph can cause "collapses" of different nodes and, therefore, the loss of hyper-extensionality of the graph itself.
With the intent of gaining a deeper understanding of the class of hereditarily finite sets, this paper investigates whether pointed hyper-extensional graphs always contain either a node or an edge whose removal does not disrupt the hyper-extensionality property.

1 Introduction

A set is hereditarily finite if it is finite and all its elements are hereditarily finite. Moreover, it is well-founded if any chain of membership relations starting from it is finite. In standard Set Theory the Extensionality axiom, establishing that two sets are equal if and only if they have the same elements, guarantees that hereditarily finite well-founded sets can be inductively constructed starting from the empty set $\emptyset$.

When also cyclic chains of memberships are allowed sets are called non-well-founded and one of the possible principles for establishing equality is Aczel's Anti-Foundation axiom based on the notion of bisimulation [1].

A hereditarily finite set S can be canonically represented through a pointed finite graph G in which each node represents a different element of the transitive closure of $\{S\}$ and the edges of G model the membership relation. Since the notion of bisimulation can be naturally defined also on graphs, this means that in the canonical representation of S there are not two different bisimilar nodes. Well-founded sets are represented by acyclic graphs, while non-well-founded sets are represented by cyclic ones (e.g., see [1] for more details).

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Now a quite general question arises: is there a natural way to inductively reason on both well-founded and non-well-founded hereditarily finite sets represented through graphs? In other terms, is there a way to inductively construct/deconstruct graphs representing hereditarily finite sets? Such question has been previously formalized and studied in [9] where the authors ask whether given the canonical representation of a set, it is always possible to find a node which can be removed producing the canonical representation of another set, i.e., without causing any bisimulation collapse. A definitive answer is not provided in [9]. In this paper we further investigate in that direction proving that there are cases in which it is not possible to remove any node without causing collapses. On the other hand, we provide positive evidence on the fact that there always exists an edge which can be safely removed.

The paper is organized as follows: Section 2 formalizes hereditarily finite sets. Section 3 relates hereditarily finite sets and pointed hyper-extensional graphs and defines keystones –elements whose removal disrupts the graph hyper-extensionality. Section 4 presents a pipeline to enumerate pointed hyper-extensional graphs. This pipeline is used in Section 5 to prove that there exist pointed hyper-extensional graphs whose nodes (edges) are all keystones. Section 6 introduces the notion of disposable element –an element whose removal does not produce collapses between nodes of the same connected components– and shows a pointed hyper-extensional graph that do not contain disposable nodes. In the same section, we prove that acyclic pointed hyper-extensional graphs always have a disposable edge and we state that this property holds also in the general case. Finally, in Section 7, we draw conclusions and suggest future works.

2 Hereditarily Finite Sets

Hereditarily finite sets are finite sets whose elements are hereditarily finite sets.

We write $\mathcal{P}(S)$ to denote the powerset of S i.e. $\mathcal{P}(S) = \{S' \mid S' \subseteq S\}$

Definition 1 (Well-founded Hereditarily Finite Sets). Well-founded hereditarily finite sets *are the elements of* $HF \stackrel{def}{=} \bigcup_{i\in\mathbb{N}} HF_i$ *where the* HF_i*'s are defined as follows:*

$$HF_i \stackrel{def}{=} \begin{cases} \emptyset & \text{if } i = 0 \\ \mathcal{P}(HF_{i-1}) & \text{otherwise} \end{cases} \tag{1}$$

Definition 2 (Non-Well-Founded Hereditarily Finite Sets). Non-well-founded hereditarily finite sets *are finite sets specified by finite systems of equations of the form:*

$$\begin{cases} Y_0 = \{X_{0,0}, \ldots, X_{0,m_0}\} \\ \quad\vdots \\ Y_n = \{X_{n,0}, \ldots, X_{n,m_n}\} \end{cases} \tag{2}$$

where $\{Y_0, \ldots, Y_n\} \supseteq \{X_{0,0}, \ldots, X_{n,m_n}\}$.

We denote the set of all the non-well-founded hereditarily finite sets by $HF^{\frac{1}{2}}$.

Let us notice that $\mathrm{HF} \subseteq \mathrm{HF}^{\frac{1}{2}}$.

3 From hereditarily finite sets to graphs

Definition 3 (Graph). *A* (directed) graph *is a tuple* (V, E) *where* V *is a finite set of nodes and* $E \subseteq V \times V$ *is a set of edges.*

If $G = (V, E)$ is a graph, we write $G \setminus e$ to indicate the graph G deprived of the edge e (i.e., $G \setminus e \stackrel{\text{def}}{=} (V, E \setminus \{e\})$) and $G \setminus v$ to denote the graphs G deprived of the node v and of all its incident edges (i.e., $G \setminus v \stackrel{\text{def}}{=} (V \setminus \{v\}, E \setminus (\{v\} \times V \cup V \times \{v\}))$). If $(v, w) \in E$, then we say that v is a *predecessor* of w and w is a *successor* of v. If there exists a sequence of nodes $v_0 \dots v_n$ such that $(v_{i-1}, v_i) \in E$ for all $i \in [1, n]$, then we say that v_n is reachable from v_0.

Definition 4 (Pointed Graph). *A graph is* pointed *if it has one node* s *such that all the nodes of the graph are reachable from* s*. We call such a node* root *of the graph.*

If we interpret the edges of a pointed graph $G = (V, E)$ as a membership relation, i.e., $(v, w) \in E$ as $w \in v$, G depicts an element of $\mathrm{HF}^{\frac{1}{2}}$. Because of this, whenever G is clear from the context, we may write $v \ni w$ in place of $(v, w) \in E$. By writing $v \ni^n w$ we mean that either $v = w$, if $n = 0$, or there exists a v' such that $v \ni v'$ and $v' \ni^{n-1} w$, if $n > 0$. If there exists a $n \in \mathbb{N}$ such that $v \ni^n w$, we can also write $v \ni^* w$.

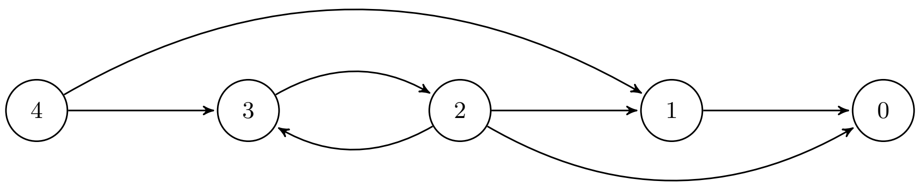

Fig. 1. This graph represents the hereditarily finite set $X_4 = \{X_3, X_1\}$ where $X_3 = \{X_2\}$, $X_2 = \{X_3, X_1, X_0\}$, $X_1 = \{X_0\}$, and $X_0 = \emptyset$.

Let us notice that two distinct pointed graphs can represent the same set. In order to map each hereditarily finite set in a single pointed graph, we need the notion of *bisimulation*.

Definition 5 (Bisimulation). *Let* $G = (V, E)$ *and* $G' = (V', E')$ *be two graphs. A* bisimulation from G to G' *is a relation* $R \subseteq V \times V'$ *such that* $(s, s') \in R$ *if and only if:*

- *for all* $s \ni p$ *there exists a* $s' \ni p'$ *such that* $(p, p') \in R$*;*
- *for all* $s' \ni p'$ *there exists a* $s \ni p$ *such that* $(p, p') \in R$.

If there exists a bisimulation R from G to G' such that $(v, v') \in R$, then we say that v and v' are bisimilar and we write $v\ {}_G{\cong}_{G'} v'$. If G and G' are the same graph, we may use the notation $\cong_G$ in place of ${}_G{\cong}_{G'}$. Whenever the graph G is clear from the context, we may also omit G from the notation $v \cong_G p$ by writing $s \cong p$. The relation $\cong$ is a bisimulation and also an equivalence relation.

We say that two graphs G and G' are bisimilar whenever there exists a bisimulation R such that for all nodes v of G there exists a node v' in G' such that $(v, v') \in R$ and vice-versa. As the equality over non-well-founded sets is defined to be bisimulation of membership graphs (Anti-Foundation Axiom [1]), two pointed graphs are bisimilar if and only if they represent the same hereditarily finite set.

Definition 6 (Collapsed Graph). *Let $G = (V, E)$ be a graph and let $[v]_{\cong}$ be the set of nodes bisimilar to v in G, i.e., $[v]_{\cong} \stackrel{def}{=} \{w \in V \mid w \cong_G v\}$. The* collapsed graph *of G is the graph $G_{\cong} \stackrel{def}{=} (V_{\cong}, E_{\cong})$ where:*

- $V_{\cong} \stackrel{def}{=} \{[v]_{\cong} \mid v \in V\}$;
- $E_{\cong} \stackrel{def}{=} \{([v]_{\cong}, [w]_{\cong}) \mid (v, w) \in E\}$.

If two graphs are bisimilar, then they share the same collapsed graph. Moreover, any graph is bisimilar to its collapsed graph. Thus, pointed collapsed graphs are a canonical form to represent hereditarily finite sets.

Definition 7 (Hyper-Extensional). *A graph G is* hyper-extensional*, or* HE*, if the only bisimulation over it is the identity, i.e., $v \cong_G v'$ implies $v = v'$.*

Since any collapsed graph is hyper-extensional, any set S in $\mathrm{HF}^{\frac{1}{2}}$ correspond to one pointed hyper-extensional graph (e.g., see [9]). Because of that, we sometime refer to pointed hyper-extensional graphs as *membership graphs*.

It is worth to underline that a membership graph may have many different roots and, thus, represent different sets. For instance, both the nodes 1 and 2 are valid roots for the graph depicted in Fig. 2.

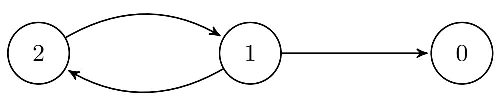

Fig. 2. Since both the nodes 1 and 2 are valid roots of the graph, this graph can represent both the hereditarily finite sets $X_2 = \{X_1\}$, where $X_1 = \{X_2, X_0\}$ and $X_0 = \emptyset$, and $X_4 = \{X_3, X_5\}$, where $X_5 = \{X_4\}$ and $X_3 = \emptyset$.

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It is easy to see that all the possible roots of a membership graph belong to the same strongly connected component.

Let us notice that nodes that share the same successors are bisimilar. Hence, if $G = (V, E)$ is a hyper-extensional graph, two nodes are the same if and only if they have the same successors. Because of that we may denote the set of the successors of $v \in V$ as v itself, i.e., $v = \{w|(v, w) \in E\}$; this is consistent with the notation $\ni$. Under the same conditions, if $v = \{v\}$ and $w = \{w\}$, then v and w are bisimilar. It follows that any hyper-extensional graph has, at most, one node v such that $v = \{v\}$ and we denote it by Ω.

Nodes or edges whose deletion causes a collapse (i.e., it reduces the number of nodes of the collapsed graph) are said *keystone*.

Definition 8 (Keystone). *Let G be a pointed hyper-extensional graph. A node n (or an edge e, respectively) of G is a* keystone *for G, if the graph $G \setminus n$ ($G \setminus e$, respectively) is not hyper-extensional.*

We are interested in establishing whether there are pointed hyper-extensional graphs in which all nodes (edges) are keystones or not. In the former case, we may identify some of them by enumerating pointed hyper-extensional graphs and testing whether all nodes (edges) are keystones.

In the following section we describe a pipeline for the enumeration of pointed hyper-extensional graphs. We used it to prove the results reported in the remaining parts of this paper.

4 Enumerating hereditarily finite sets

The enumeration of all the pointed hyper-extensional graphs up to a given order n –having n nodes– is inherently exponential with respect to $n^2/2$. As a matter of fact, as n grows, the number of acyclic graphs having order n tends to $2^{\binom{n}{2}}/M\sigma^n$, where $M \approx 0.57436$ and $\sigma \approx 1.48807$ [2,3]. Moreover, roughly 32.6% of these graphs are (hyper-)extensional [16]. It follows that enumerating acyclic (hyper-)extensional graphs having order n lays in the time complexity class $\Omega(2^{n^2/2})$.

In order to produce all the pointed hyper-extensional graphs, we could both generate all the directed graphs and retain only those that are hyper-extensional and pointed[3]. Unfortunately, there are 2^{n^2} directed graphs of order n –having n nodes– and the large part of them are not even connected.

A significative improvement for this strategy was obtained by observing that both the properties of being hyper-extensional and pointed are preserved under isomorphism. Thus, either all graphs in an isomorphic class –the class of all the graphs that are pairwise isomorphic– are hyper-extensional or they are all not hyper-extensional. This also holds for the property of being pointed.

The isomorphic classes of directed graphs has been extensively studies (e.g., see [12,4,8,6]). They still are super-exponential in number with respect to the order n of the investigated graphs, (in particular, they are at least $2^{n^2}/n!$), but their abundance grows significantly slower than 2^{n^2}. For instance, for $n = 6$, 32,

[3] Let us notice that this is not an enumeration for pointed hyper-extensional graphs since two bisimilar graphs which are not isomorphic can be retain.

and 64 each class of isomorphic graphs contains in average more than 44595, 10^{43}, and 10^{108} elements, respectively (see [13]).

We implement a pipeline to enumerate all the pointed hyper-extensional graphs in `SAGE` [15]. A representative for each of the isomorphic classes is produced by using the command `canaug_traverse_edge(...)`. The pipeline should retain a graph only if it is hyper-extensional and pointed. In order to reduce the average time required to test these properties, two preliminary heuristics are applied. Since pointed graphs have at most one source –node that has no incoming edges–, the first heuristic filters graphs that have more than one source. We also noticed that all hyper-extensional pointed graphs, but the one representing Ω, must include exactly one sink node –node with no outgoing edges–, i.e., $\emptyset$. Thus, among the graphs that have survived the first filter, the pipeline considers exclusively the ones that either have one sink or that have no sinks and one node; the latter case correspond to Ω.

The next step is to identify pointed graphs. Given a graph G, the command `strongly_connected_components_digraph(`G`)` produces a new graph G', analogous to it, in which each strong connected component of G has been collapsed to a distict node. The resulting graph is pointed if and only if the original one is pointed too. Moreover, as G' does not contain non-trivial strongly connected components, if it is pointed, then it must have exactly one source. Hence, it is possible to decide whether G is pointed or not by both testing the existence of one single source in G' and, if this is the case, by performing a reachability computation from it.

Finally, the pipeline verifies hyper-extensionality of each of the remaining graphs by computing its maximum bisimulation [7,10]. If no pairs of nodes are bisimilar, the considered graph is pointed and hyper-extensional and, thus, it is kept.

Since all valid roots of a membership graph belong to the same strongly connected component, our pipeline is also able to compute the number of hereditarily finite sets that are represented by graphs of a given order. In particular, the hereditarily finite sets that have each of the issued graphs $\mathcal{G}$ as membership graph are in number as many as the nodes of the strongly connected components that contains a root for $\mathcal{G}$ itself.

Table 1 lists, for each order up to 5, the number of isomorphic classes of directed graphs (with self-loops), the number of pointed hyper-extensional graphs, and the number of hereditarily finite sets as they are computed by our pipeline. The same table also details the number of well-founded hereditarily finite sets that is reported in [14].

5 Do non-keystone always exist?

In [9], it has been proved that if G is pointed hyper-extensional and acyclic (i.e., it represents a well-founded set), then not all the nodes of G are keystones. The more challenging case of cyclic graphs was left open.

Our pipeline, described in Section 4, can be used to produce all the pointed hyper-extensional graphs having up to 5 nodes. We test the existence of a non-keystone node in them by removing each of the nodes and by testing hyper-extensionality of the resulting graph. None of the considered graphs contains exclusively keystone nodes. Computing all the pointed hyper-extensional graphs of order 6 is too time consuming. However, by using the above method, we have discovered a graph in which all nodes are keystones and, as a consequence, we prove the following theorem.

Theorem 1. *There exists a non-empty pointed hyper-extensional graph such that all of its nodes are keystones.*

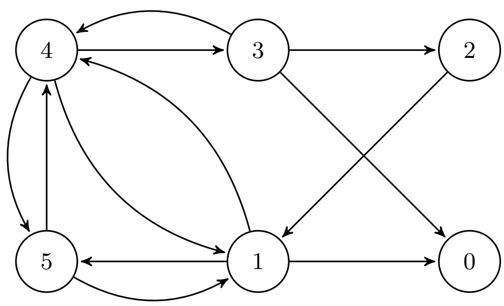

Fig. 3. A pointed hyper-extensional graph whose nodes are all keystones.

Proof. Let us consider the graph G_0 depicted in Fig. 3. All of its nodes, but 0, belong to the same strongly connected component (i.e. there is a path from v to w for all $v \neq 0$ and $w \neq 0$) and 0 is reachable from both 1 and 3. It follows that G_0 is pointed. Moreover, G_0 is hyper-extensional. However, $1 \cong_{G_0 \setminus 0} 2$, $0 \cong_{G_0 \setminus 1} 2$, $4 \cong_{G_0 \setminus 2} 5$, $4 \cong_{G_0 \setminus 3} 5$, $1 \cong_{G_0 \setminus 4} 3$, and $1 \cong_{G_0 \setminus 5} 3$. This proves the claim.

As far as keystone edges are concerned, we easily prove the following result.

Theorem 2. *There exists a pointed hyper-extensional graph such that it has at least one edge and all of its edges are keystones.*

Proof. Let us consider the graph G_1 depicted in Fig. 4. It is possible to reach 0 from 1, thus, it is pointed. Moreover, since 0 does not reach any node, G_1 is hyper-extensional. However, $G_1 \setminus (1,0)$ contains no edges and, hence, $0 \cong_{G_1 \setminus (1,0)} 1$. This proves the claim.

Our pipeline highlights that, up to order 5, chains –connected graphs whose nodes, but $\emptyset$, have one single successor– are the only pointed hyper-extensional graphs whose edges are all keystones (see Table 1). This leads us to consider separately each of connected components produced by the elimination of a graph

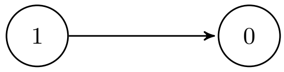

Fig. 4. A pointed hyper-extensional graph whose edges are all keystones.

element. In the following section, we investigate whether there always exists a graph element whose elimination generates connected components that, individually, are hyper-extensional.

6 Hyper-extensionality and weak connected components

As a first step, we need to formalize the notion of connectivity over directed graphs. Such a notion, called *weak connectivity*, coincides with the connectivity over the corresponding undirected graph.

Definition 9 (Weak connectivity). *A graph* (V, E) *is* weakly connected *if, for any* $V' \subsetneq V$*, there exists an edge* $e \in E$ *such that either* $e \in V' \times (V \setminus V')$ *or* $e \in (V \setminus V') \times V'$*.*

Definition 10 (Disposable). *Let* G *be a pointed hyper-extensional graph. A node* n *(or an edge* e*, respectively) of* G *is* disposable *whenever the weakly connected components of* $G \setminus n$ *(or* $G \setminus e$*, respectively) are hyper-extensional graphs.*

Proposition 1. *If* v *is not disposable for* G*, then it is a keystone for it.*

Proof. If v is not disposable, then there exists a weakly connected component of $G \setminus v$ that is not hyper-extensional and there exists a bisimulation that is not the identity for it. By extending this bisimulation with the identity over the other weakly connected components, we obtain a bisimulation for $G \setminus v$ that is not the identity. Thus, $G \setminus v$ is not hyper-extensional and v is a keystone for G. □

The graph depicted in Fig. 3 proves the following result.

Theorem 3. *There exists a non-empty pointed hyper-extensional graph such that none of its nodes is disposable.*

Proof. Let us consider the graph $G_0 = (V_0, E_0)$ depicted in Fig. 3. As already observed above it is hyper-extensional and pointed. Moreover, all of its nodes are keystones by Theorem 1 and $G_0 \setminus v$ is weakly connected for all $v \in V_0$. It follows that none of its nodes is disposable. □

As far as disposable edges are concerned, we preliminarily prove that any pointed hyper-extensional acyclic graph has a disposable edge. First of all, we need to introduce the notion of *rank*.

				Only keystone		No disposable	
Order	IC	HP	$\widetilde{\mathrm{HF}}^{\frac{1}{2}}$	nodes	edges	nodes	edges
1	2	2	2(1)	0(0)	0(0)	0(0)	1(1)
2	10	2	2(1)	0(0)	1(1)	0(0)	0(0)
3	104	12	16(2)	0(0)	1(1)	0(0)	0(0)
4	3044	252	504(9)	0(0)	1(1)	0(0)	0(0)
5	291968	18439	52944(88)	0(0)	1(1)	0(0)	0(0)

Table 1. Number of pointed hyper-extensional graphs whose nodes/edges are all keystones and whose nodes/edges are all disposable. The columns labelled as IC, HP, and $\widetilde{\mathrm{HF}}$ report the number of isomorphic classes of directed graphs (with self-loops), of pointed hyper-extensional graphs, and of hereditarily finite sets (roots of the pointed hyper-extensional graphs), respectively. All the data concerning the well-founded domain are reported in brackets. The number of well-founded hereditarily finite sets (pointed hyper-extensional acyclic graphs) is taken from [14]

Definition 11 (Rank [5]). *Let $G = (V, E)$ be an acyclic graph and let v be one of its nodes. The* rank *of v in G, $rank(G, v)$, is defined as follows:*

$$rank(G, v) \stackrel{def}{=} \begin{cases} 0 & \textit{if } v \textit{ is a sink} \\ 1 + \max\limits_{(v,u)\in E}(rank(G, u)) & \textit{otherwise} \end{cases}$$

Notice that, whenever G is acyclic, all the paths from v to a sink are finite in length and $rank(G, v)$ is well-defined.

It is easy to see that if G is pointed and acyclic, then it has one single root and its rank is greater than those of the other nodes of G.

Lemma 1. *If $G = (V, E)$ is a pointed acyclic graph, then it has one single root p and $rank(G, p) > rank(G, q)$ for all $q \in V \setminus \{p\}$.*

Proof. If G had two roots, p_1 and p_2, then, by definition of root, both p_1 is reachable from p_2 and p_2 is reachable from p_1. It follows that G is cyclic and this contradicts our hypothesis. Hence, G must have one root p.

Let us assume that there exists $q \neq p$ such that $rank(G, p) \leq rank(G, q)$. Since q is reachable from p, there exists a finite sequence of nodes $p_0, p_1, \ldots p_h$ such that $p_0 = q$, $p_h = p$, and $p_{i+1} \ni p_i$ is an edge of G for all $i \in [0, h-1]$. Since $p \neq q$, h should be greater than 0. Moreover, by induction on i, we can prove that $rank(G, p_i) \geq rank(G, q) + i$. It follows that $rank(G, p)$ should be greater or equal to $rank(G, q) + h$ where $h \in \mathbb{N} \setminus \{0\}$. However, this contradicts our hypothesis and, thus, it proves that all the nodes in $V \setminus \{p\}$ must have a rank smaller than that of p. □

The following lemmas relate the ranks of bisimilar nodes and show how bi simulations are affected by edge removals.

Lemma 2 ([5]). *Let G be an acyclic graph and let v and u be two of its nodes. If $v \cong u$, then $rank(G, v) = rank(G, u)$.*

Lemma 3. *Let G be an acyclic graph, let (v, u) be one of its edges, and w one of its nodes. If $rank(G, w) < rank(G, v)$, then $w\ {}_{G'}\!\cong_G w$ where $G' = G \setminus (v, u)$.*

Proof. We prove the thesis by induction on the rank of w.

$\mathbf{rank(G, w) = 0}$ If $rank(G, w) = 0$, then w is a sink in G. Hence, it is a sink also in G' and $w\ {}_{G'}\!\cong_G w$.

$\mathbf{rank(G, w) > 0}$ Let us assume that $q\ {}_{G'}\!\cong_G q$ for all node q in G such that $rank(G, q) < rank(G, w)$. From the definition of rank, it follows that $q\ {}_{G'}\!\cong_G q$ for all $w \ni q$ in G. Since $rank(G, w) < rank(G, v)$, $v \neq w$ and $v \ni u \neq w \ni q$ for all edge $w \ni q$ in G. Hence, by definition of bisimulation, $w\ {}_{G'}\!\cong_G w$. □

Lemma 2 implies that two nodes that have different ranks are not bisimilar, while from Lemma 3 it follows that edge removal can collapse only nodes whose ranks are greater or equal to that of the edge source. Since Lemma 1 proves that roots have maximum rank in pointed acyclic graphs, removing an edge outgoing from the roots avoids collapses in other nodes.

Theorem 4. *Any pointed hyper-extensional acyclic graph that has at least one edge has a disposable edge.*

Proof. Let G be a pointed hyper-extensional acyclic graph and let p the root of G. If p has only one outgoing edge, then we can safely remove it and we get two distinct weakly connected components: one of them contains only p, while, in the other one, there are no collapses, since no outgoing edges have been removed. Thus, the removed edge is disposable in G.

If p has at least two outgoing edges, then let $p \ni q$ be an edge of G such that $rank(G, q) = \min_{p \ni p'}(rank(G, p'))$. We prove that $p \ni q$ is disposable. Since p has at least two outgoing edges, we can conclude that $rank(G', p) = rank(G, p)$, where $G' = G \setminus (p, q)$, by definition of rank. By Lemmas 1 and 2, no node $w \neq p$ collapses with p in G'. Moreover, $w\ {}_{G'}\!\cong_G w$ for all $w \neq p$ by Lemmas 3 and 1. If two nodes w, w' collapsed in G', then $w'\ {}_{G}\!\cong_{G'} w'\ {}_{G'}\!\cong_{G'} w\ {}_{G'}\!\cong_G w$. Hence, they should collapse also in the original graph G which contradicts the hyper-extensionality of G. Thus, no collapses occur in G' and $p \ni q$ is disposable. □

We are confident that this result can be generalized also to cyclic graphs (i.e., non-well-founded sets). As shown in Table 1 up to order 5 the only hyper-extensional pointed graph without disposable edges is the one without edges.

7 Conclusions

In this paper we considered the problem of removing parts from an hereditarily finite set without causing bisimulation collapses. We exploited a `SAGE` pipeline that we implemented to prove that the problem has a negative answer if one is interested in removing a node. On the other hand, a positive answer is expected in the case of removal of one edge, provided that only weakly connected nodes are compared.

The problem considered in this paper have been previously studied in [9] and a positive solution would allow to extend the results described in [11] to non-well-founded sets.

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